Aerodynamic Symmetry of Aircraft and Guided Missiles

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A technique is developed that takes advantage of the inherent configurational symmetries of aircraft and guided missiles to eliminate some force and moment derivatives. Starting with the Principle of Material Indifference, tensor analysis is employed to derive two simple conditions for vanishing aerodynamic derivatives. The results apply to derivatives of arbitrary order, taken with respect to linear and angular velocities, linear accelerations, and control surface deflections. Two charts are presented that sift out the vanishing derivatives up to second order for missiles with tetragonal symmetry, and up to third order for aircraft with reflectional symmetry.

Introduction

MODERN aircraft and missile dynamics require sophisticated mathematical modeling of aerodynamic forces. Consider, for instance, the simulation of guided tactical missiles. The requirement that they accurately home into a target demands great maneuverability, resulting in large angles-of-attack and control surface deflections. The linear representation of the aerodynamic forces, in general, is not justified except for preliminary design work. Accurate simulations of the airframe dynamics are only possible if higher order aerodynamic effects are included. The electronic computer becomes an indispensable tool in modeling the aerodynamic forces by tabular functions. However, restrictions for storage and computer time require that the number of independent variables in the aerodynamic tables be kept to a minimum. The dependence on the other variables is then expressed analytically by a Taylor series expansion. The coefficients of the series are the aerodynamic derivatives. They are given in tabular form as functions of the implicit variables. Their number can become quite large if second and third order effects are included.

Fortunately, all flight vehicles possess certain configurational symmetries that reduce the number of derivatives. Maple and Synge[1] investigated the vanishing of aerodynamic derivatives in the presence of rotational and reflectional symmetries. They considered the dependence of the aerodynamic forces on linear and angular velocities only. Complex variables were used to derive the results. Because of the two-dimensional form of complex variables, only the cross velocities and aerodynamic coefficients can be replaced by concise complex formulations. The longitudinal velocities and coefficients require separate treatment. After expanding the aerodynamic coefficients in a complex Taylor series and after imposing the symmetry conditions, the derivatives must be converted back into the real domain.

The Maple-Synge Theory contributed to the solution of many nonlinear ballistic problems 20 to 25 years ago. However, with the advent of guided missiles, the dependence of the aerodynamic forces on unsteady flow effects and control surface deflections has increasingly gained in importance. The Maple-Synge Theory does not address these dependencies, and its application is burdened by complicated rules.

The object of the report is to derive simple conditions that determine, for aircraft and guided missiles, the vanishing of the aerodynamic derivatives due to reflectional or tetragonal (90 deg rotational) symmetries. The aerodynamic forces are assumed to be functions of linear and angular velocities, linear accelerations, and control surface deflections. The derivation is based on the general Principle of Material Indifference and is rigorously executed using tensor analysis.

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Aerodynamic Derivatives

In this report, the aerodynamic forces are assumed to depend on the state of the fluid medium, the external configuration, the linear velocity and acceleration, and the angular velocity

$$y_i = f_i\{p,\rho,\ell_m^{(\tau)}, n_m^{(\tau)}, v_m, \omega_m, \dot{v}_m, \delta_m\} \tag{1}$$

where y_i is the six-dimensional force vector. Its components in body axes are

$$y_i = [X,Y,Z,L,M,N];\ i=1,\ldots,6 \tag{2}$$

composed of the force components X, Y, Z parallel to the x, y, z axes and the moment components L, M, N about the same three axes. The term, "aerodynamic forces," will be used in the generalized sense to refer to the aerodynamic forces and moments. Subscript and matrix notation is used interchangeably. The subscript notation follows tensor calculus conventions. Brackets enclose matrix elements. The components of a vector, see Eq. (2), are displayed in the transposed form with commas separating the elements for clarification. The six-dimensional vector valued functional of the aerodynamic forces f_i depends on the state of the fluid as expressed by the scalar quantities: pressure p and density ρ. The external configuration is formulated by a family of three-dimensional position vectors $\ell_m^{(\tau)}$, which locate the τ numbered surface elements, and by the corresponding three-dimensional normal surface vectors $n_m^{(\tau)}$. The deflections of the control surfaces, causing a change in the external configuration, are represented by a three-dimensional vector. Its components in body axes are

$$\delta_m = [\delta p, \delta q, \delta r];\ m=1,2,3 \tag{3}$$

where δp, δq, δr generate roll, pitch, and yaw moments, respectively. The motions of the vehicle relative to the air are given by the following three-dimensional vectors with their respective components in body axes

Linear velocity

$$v_m = [u,v,w] \tag{4}$$

Angular velocity

$$\omega_m = [p,q,r] \tag{5}$$

Linear acceleration

$$\dot{v}_m = [\dot{u}, \dot{v}, \dot{w}] \tag{6}$$

Usually, the functional dependence that the aerodynamic forces have on the state of the fluid medium and the external configuration are implicitly included in the functional. With the remaining explicit variables abbreviated by the 12-dimensional state vector

$$z_j = [u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}, \delta p, \delta q, \delta r] \tag{7}$$

Eq. (1) assumes the shorter form

$$y_i = d_i\{z_j\}; i = 1, 2, \ldots, 6; j = 1, \ldots, 12 \tag{8}$$

The aerodynamic force functional is expanded in a Taylor series in terms of the 12 state variable components z_j relative to a reference state $\bar{z}_j$. The Taylor series expansion is mathematically justified, if the partial derivatives in the expansion are continuous and the expansion variables $\epsilon z_j = z_j - \bar{z}_j$ are small. For aircraft and missiles, the aerodynamic forces are continuous functions of their states for most flight maneuvers. However, unsteady effects, such as vortex shedding, can introduce discontinuities that cannot be presented accurately by Taylor expansions. In subscript notation, the Taylor series assumes the form

$$\begin{aligned} y_i = d_i\{\bar{z}_j\} &+ (\partial d_i/\partial z_{j_1})\epsilon z_{j_1} + (\partial^2 d_i/2\partial z_{j_1}\,\partial z_{j_2})\epsilon z_{j_1}\epsilon z_{j_2} + \ldots \\ &+ (\partial^k d_i/k!\partial z_{j_1}\ldots\partial z_{j_k})\epsilon z_{j_1}\ldots\epsilon z_{j_k} + \ldots \\ &i = 1, \ldots, 6; j_1 = \ldots = j_k = 1, \ldots, 12 \end{aligned} \tag{9}$$

The coefficients of the expansion are the aerodynamic derivatives, evaluated at the reference state $\bar{z}_j$. The kth derivative is a $k+1$ order tensor. It is abbreviated by

$$\partial^k d_i/\partial z_{j_1}\partial z_{j_2}\ldots\partial z_{j_k} = D_i{}^{j_1 j_2 \ldots j_k} \tag{10}$$

and is a function of the implicit variables p, ρ, $\ell_m^{(\tau)}$, $n_m^{(\tau)}$. As an example, the third order derivative with $i = 4$, $j_1 = 1$, $j_2 = 5$, $j_3 = 11$, expressed in body axes, reads

$$\partial^3 d_4/\partial z_1 \partial z_5 \partial z_{11} = D_4{}^{1\,5\,11} = \partial^3 L/\partial u \partial q \partial \delta q = L_{uq\delta q} \tag{11}$$

Configurational Symmetries

Most aircraft and guided missiles have a planar or cruciform external shape. The planar configuration dominates among aircraft and cruise missiles, whereas missiles that require rapid terminal maneuverability have cruciform airframes. Two types of symmetries are therefore considered: reflectional and tetragonal (90 deg rotational).

To derive the conditions for vanishing aerodynamic derivatives, precise definitions of these symmetries are required. In the case of reflectional symmetry, the existence of a plane, satisfying certain conditions, is required, whereas tetragonal symmetry calls for an axis with specific characteristics.

First, a formulation of the external configuration will be given. The external shape of a body is described by N surface elements $A^{(\tau)}$, where N is an integer as large as necessary. Each surface element is located by the position vector $\ell_m^{(\tau)}$ and oriented by its unit normal vector $n_m^{(\tau)}$, $m = 1, 2, 3$. Specific surface elements are designated by the superscripts λ and δ with their respective position vectors $\ell_m^{(\lambda)}$, $\ell_m^{(\delta)}$ and unit normal vectors $n_m^{(\lambda)}$, $n_m^{(\delta)}$.

By definition, a body possesses *reflectional symmetry* if there exists a plane with the characteristics that all of the position vectors $\ell_m^{(\tau)}$ and all of the unit normal vectors $n_m^{(\tau)}$ which originate at every point of the plane occur in pairs, related by

$$\ell_m^{(\lambda)} = H_{mp}\ell_p{}^{(\delta)} \tag{12}$$

$$n_m^{(\lambda)} = H_{mp}n_p{}^{(\delta)} \tag{13}$$

H_{mp} is called the reflection tensor. Its components in body coordinates with the x and z axes subtended in the symmetry plane are

$$H_{mp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

The reflection tensor leaves the x and z components of a vector unchanged, but reverses the sign of the y component. It executes an improper rotation because its determinant is negative, that is, $\det(H) = -1$.

Particular attention needs to be directed to the operation of the reflection tensor on the so-called axial vectors. Examples of axial vectors are the angular velocity ω_m, the surface deflection δ_m, and the aerodynamic moment vector. Mathematically, axial vectors are obtained by contracting skew-symmetric tensors with the permutation symbol ϵ_{mnp}

$$\omega_m = 0.5\epsilon_{mnp}\Omega_{np} \tag{15}$$

To derive the operation of the reflection tensor on an axial vector, first consider the operation on its skew-symmetric tensor form

$$\Omega'_{np} = H_{nq}\Omega_{qr}H_{pr} \tag{16}$$

and contract both sides by ϵ_{mnp}

$$\epsilon_{mnp}\Omega'_{np} = \epsilon_{mnp}H_{nq}H_{pr}\Omega_{qr} \tag{17}$$

From Jeffreys[2], the right hand side of Eq. (17) reduces to

$$\epsilon_{mnp}H_{nq}H_{pr}\Omega_{qr} = \det(H)\,H_{mt}\epsilon_{tqr}\Omega_{qr} \tag{18}$$

because $\det(H) = -1$, Eq. (17) becomes

$$\epsilon_{mnp}\Omega'_{np} = -H_{mt}\epsilon_{tqr}\Omega_{qr} \tag{19}$$

Dividing Eq. (12) by two and introducing the definition of an axial vector, Eq. (15), give the desired relationship

$$\omega'_m = -H_{mt}\omega_t \tag{20}$$

In contrast to regular vectors, the operation of the reflection tensor on an axial vector leaves the y component unchanged, but reverses the sign of the x and z components.

The 12-dimensional aerodynamic state vector, defined by Eq. (7), is partitioned into four three-dimensional vectors, alternating between regular and axial vectors

$$z_j = [v_m, \omega_m, \dot{v}_m, \delta_m]; j = 1, \ldots, 12; m = 1, 2, 3 \tag{21}$$

If the state vector is subjected to the reflection tensor, each of the partitioned vectors is operated on by H_{mt}

$$[v'_m, \omega'_m, \dot{v}'_m, \delta'_m] = [H_{mt}v_t, -H_{mt}\omega_t, H_{mt}\dot{v}_t, -H_{mt}\delta_t] \tag{22}$$

Expressed in shorter form, Eq. (22) becomes

$$z_j' = H_{jf}z_f; j = f = 1, \ldots, 12 \tag{23}$$

The 12-dimensional reflection tensor H_{jf} is a diagonal matrix in body coordinates

$$H_{jf} = \mathrm{diag}(1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,) \quad (24)$$

The alternating signs of the unit diagonal elements suggest an even more compact formulation of Eq. (23)

$$z_j' = (-1)^{j+1} z_j \quad (25)$$

The six-dimensional aerodynamic force vector, defined by Eq. (2), is partitioned into one regular and one axial vector. If subjected to the reflection tensor, relationships similar to those of Eqs. (23-25) hold

$$y_i' = H_{ih} y_h; i=h=1,\ldots,6 \quad (26)$$

$$H_{ih} = \mathrm{diag}(1,-1,1,-1,1,-1) \quad (27)$$

$$y_i' = (-1)^{i+1} y_i \quad (28)$$

The considerable simplifications which Eqs. (25) and (28) exhibit will be responsible for the simple formulation of the conditions for vanishing aerodynamic derivatives.

Finally, the definition of tetragonal symmetry will be given. A body possesses *tetragonal symmetry* if there exists an axis with the characteristics that all of the position vectors $\ell_m^{(\tau)}$, originating at every point of the axis, and all of the unit normal vectors $n_m^{(\tau)}$ of the surface elements occur in pairs as related by

$$\ell_m^{(\lambda)} = T_{mp} \ell_p^{(\delta)} \quad (29)$$

$$n_m^{(\lambda)} = T_{mp} n_p^{(\delta)} \quad (30)$$

The components of the tetragonal symmetry tensor T_{mp} in body coordinates, with the x axis coinciding with the symmetry axis, are

$$T_{mp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (31)$$

The tetragonal symmetry tensor leaves the x components unaltered, but exchanges the y and z components with sign reversal of the components. It executes a proper rotation because its determinant is positive and has the value one. It follows from Eq. (18) that the tetragonal symmetry tensor operates on axial vectors just as it does on regular vectors.

Principle of Material Indifference

The derivation of the conditions for vanishing aerodynamic derivatives is founded on the physical Principle of Material Indifference (PMI). The PMI has been recognized as early as 1903 by Zaremba and in 1906 by Jauman (see the historical expose by Truesdell and Noll[3]). The PMI simply states that constitutive equations are invariant under spatial rigid rotations. Noll[4] has provided a precise mathematical formulation. Applied to the aerodynamic problem at hand, the PMI asserts that the physical process of generating aerodynamic forces is independent of spatial attitude.

A mathematical formulation will now be given. The process of generating the aerodynamic forces y_i (for example, in a particular flight test), is described by the vector valued functional f_i of the state variables x_j

$$y_i = f_i\{x_j\} i=1,\ldots,6; j=1,2,3,\ldots \quad (32)$$

Equation (1) may serve as an example. Next, consider the same process at another spatial attitude (for example, the same flight test at a different geographical location). The variables associated with this second test are designated by a prime. The state variables of the two tests are related by the orthogonal rigid rotation tensor R_{jp}

$$x_j' = R_{jp} x_p \quad (33)$$

From the PMI, it follows that the functional relationships of the two tests are the same. The aerodynamic forces of the second test are therefore

$$y_i' = f_i\{x_j'\} \quad (34)$$

and are related to the forces of the first test by the same rotation tensor

$$y_i' = R_{im} y_m \quad (35)$$

Substitution of Eqs. (32) and (34) into Eq. (35) results in

$$R_{in} f_n\{x_j\} = f_i\{x_j'\} \quad (36)$$

Finally, Eq. (33) substituted into Eq. (36) yields the mathematical formulation of the PMI applied to the aerodynamic force generating process

$$R_{in} f_n\{x_j\} = f_i\{R_{jp} x_p\} \quad (37)$$

Equation (37) reads from left to right: the vector valued function f_n of the state vector x_j, rotated through the rigid rotation R_{in}, equals the same vector valued function of the state variables rotated through the same tensor R_{jp}. A functional with the property expressed by Eq. (37) is called an isotropic function. The rotation is allowed to be proper or improper; i.e., its determinant can be plus or minus one.

An important relationship between the aerodynamic derivatives of the same two flight tests, related by a rigid rotation, shall now be derived by employing the PMI. Equation (32) is expanded in a Taylor series about the reference state $\bar{x}_j$ in terms of the small variables $\epsilon x_j = x_j - \bar{x}_j$

$$y_i = f_i\{\bar{x}_j\} + (\partial f_i / \partial x_{j_1}) \epsilon x_{j_1} + \ldots$$
$$+ (\partial^k f_i / k! \partial x_{j_1} \ldots \partial x_{j_k}) \epsilon x_{j_1} \ldots \epsilon x_{j_k} + \ldots \quad (38)$$

and, similarly, Eq. (34) is expanded

$$y_i' = f_i\{\bar{x}_j'\} + (\partial f_i / \partial x_{j_1}') \epsilon x_j{}'_1 + \ldots$$
$$+ (\partial^k f_i / k! \partial x_{j_1}' \ldots \partial x_{j_k}') \epsilon x_{j_1}' \ldots \epsilon x_{j_k}' + \ldots \quad (39)$$

Note that in both cases, the partial derivatives have the same functional form of the state variables x_j and x_j', respectively. They are tensors of order $k+1$. Substituting Eqs. (38) and (39) into Eq. (35) and replacing the primed state vector by Eq. (33) yield the desired relationships by comparing terms of equal powers in x_j

$$R_{in} f_n\{\bar{x}_j\} = f_i\{\bar{x}_j'\} \quad (40a)$$

$$R_{in}(\partial f_n / \partial x_{j_1}) = /\partial f_i (\partial x_{r_1}') R_{r_1 j_1} \quad (40b)$$

$$R_{in}(\partial^k f_n / \partial x_{j_1} \ldots \partial x_{j_k})$$
$$= (\partial^k f_i / \partial x_{r_1}' \ldots \partial x_{r_k}') R_{r_1 j_1} \ldots R_{r_k j_k} \quad (40c)$$

$$i=n=1,\ldots,6; j_1=\ldots=j_k=r_1=\ldots=r_k=1,2,3,\ldots$$

Planar Vehicles

The PMI shall now be applied to the aerodynamic derivatives of vehicles with reflectional symmetry. Equation (1) may describe the aerodynamic forces of a particular flight test. Consider a second flight test under the same conditions, but related through the reflection tensor H_{mt}. The aerodynamic forces are obtained from the PMI, according to Eq. (34) and in view of Eq. (33)

$$y_i' = f_i\{p, \rho, H_{mt}\ell_t^{(\tau)}, H_{mt}n_t^{(\tau)}, H_{mt}v_t, -H_{mt}\omega_t, H_{mt}\dot{v}_t, -H_{mt}\delta_t\} \quad (41)$$

$$i = 1, \ldots, 6; m = t = 1,2,3$$

The scalars p, ρ are invariants under the operation of the reflection tensor. Also, because the vehicle possesses reflectional symmetry, the operations of H_{mt} on the position vectors $\ell_t^{(\tau)}$ and the unit normal vectors $n_t^{(\tau)}$ are invariants by Eqs. (12) and (13). That is, in the two flight tests, the external configuration of the vehicle appears unaltered. The invariants are absorbed in the functional. Then, with the abbreviations of Eqs. (7, 8, 22, 23) the two flight tests are described by

$$y_i = d_i\{z_j\} \quad (42)$$

$$y_i' = d_i\{H_{jf}z_f\} = d_i\{z_j'\} \quad (43)$$

$$i = 1, \ldots, 6; j = f = 1, \ldots, 12$$

Both aerodynamic force functional of Eqs. (42) and (43) are expanded in the Taylor series, such as Eq. (9), and in a similar expansion with primed variables. They have the same form as the Eqs. (38) and (39). The PMI for aerodynamic derivatives, Eq. (40), can be applied directly by substituting the general rotation tensor R_{if} for the specific reflection tensor H_{if}

$$H_{in}(\partial^k d_n/\partial z_{j_1} \ldots \partial z_{j_k}) = (\partial^k d_i/\partial z'_{r_1} \ldots \partial z'_{r_k}) H_{r_1 j_1} \ldots H_{r_k j_k} \quad (44)$$

$$i = n = 1, \ldots, 6; j_1 = \ldots = j_k = r_1 = \ldots r_k = 1, \ldots, 12$$

In body coordinates, the reflection tensor has the diagonal form of Eq. (24). With the more compact notation of Eq. (10) and with the sign functional Eq. (25) replacing H_{in} and $H_{r_1 j_1} \ldots H_{r_k j_k}$, Eq. (44) can be written

$$(-1)^{i+1} D_i^{j_1 \ldots j_k} = D_i^{j_1 \ldots j_k} (-1)^{j_1+1} \ldots (-1)^{j_k+1} \quad (45)$$

Collecting all the sign functionals yields the major result of this report

$$D_i^{j_1 \ldots j_k} = (-1)^{\Sigma j_k + k + i + 1} D_i^{j_1 \ldots j_k} \quad (46)$$

Equation (46) states *that aerodynamic derivatives* $D_i^{j_1 \ldots j_k}$ *of a vehicle with reflectional symmetry vanish if the sum* $\Sigma j_k + k + i + 1$ *is an odd number.* When the exponent of (-1) is odd, a negative sign will appear at the right hand side of Eq. (46), and the same two derivatives with different signs can only be equal if their values are zero. The subscript i indicates the force components and the superscripts $j_1 \ldots j_k$ fix the components of the state vector of the first, second, . . ., kth partial derivatives. The condition for vanishing derivatives, as formulated by Eq. (46), greatly simplifies the mathematical modeling of the nonlinear behavior of aerodynamic forces; it treats any order derivative with equal ease and applies to unsteady flow effects and control effectiveness.

Cruciform Vehicles

To derive the condition for vanishing aerodynamic derivatives of vehicles with tetragonal symmetry, use is made of the fact that a cruciform vehicle has two planes of reflectional symmetry. The two planes are rotated into each other by the tetragonal symmetry tensor T_{pi}, and they intersect at the axis of symmetry. The PMI is applied twice relative to the two symmetry planes. The first application of the PMI was carried out for, say, flight tests 1 and 2 in the previous section. From the two flight tests, Eqs. (42) and (43), related by the reflectional tensor, the condition for vanishing derivatives, Eq. (46), was obtained. This is the first condition that eliminates aerodynamic derivatives of vehicles with tetragonal symmetry. The second condition is derived from the application of the PMI to say, flight tests 3 and 4, which are generated from flight tests 1 and 2 of the previous section by a 90 deg rigid rotation T_{pi}

$$T_{pi}y_i = T_{pi}d_i\{T_{qj}z_j\} \quad (47)$$

$$T_{pi}y_i' = T_{pi}d_i\{T_{qj}z_j'\} \quad (48)$$

Because tests 1 and 2 are mirror images, so are tests 3 and 4, with the mirror lying in the second plane of symmetry. As in the case of the first two tests, the PMI is applied to tests 3 and 4. To maintain the similiarity, new variables are introduced in Eqs. (47) and (48)

$$t_p = c_p\{s_q\} \quad (49)$$

$$t_p' = c_p\{s_q'\} \quad (50)$$

The definitions of the new variables follow by comparison with Eq. (47) and (48). Both equations are expanded in the Taylor series. They again have the form of Eqs. (38) and (39), and the PMI for aerodynamic derivatives, Eq. (40), can be applied with the general rotation tensor R_{pn} replaced by the specific reflection tensor H_{pn}

$$H_{pn}(\partial^k c_n/\partial s_{q_1} \ldots \partial s_{q_k}) = (\partial^k c_p/\partial s'_{r_1} \ldots \partial s'_{r_k}) H_{r_1 q_1} \ldots H_{r_k q_k} \quad (51)$$

With an abbreviation equivalent to Eq. (10) and the introduction of sign functionals as in the latter part of the previous section, the more compact formulation is obtained

$$C_p^{q_1 \ldots q_k} = (-1)^{\Sigma q_k + k + p + 1} C_p^{q_1 \ldots q_k} \quad (52)$$

Equation (52) is the second condition for the vanishing of aerodynamic derivatives of cruciform vehicles. To be of practical use, a relationship between the $C_p^{q_1 \ldots q_k}$ and $D_i^{j_1 \ldots j_k}$ derivatives must be established. Take the kth partial derivative with respect to s_j on the right hand side of Eqs. (47) and (49)

$$\partial^k c_p/\partial s_{q_1} \ldots \partial s_{q_k} = T_{pi}(\partial^k d_i/\partial z_{r_1} \ldots \partial z_{r_k}) T_{q_1 r_1} \ldots T_{q_k r_k} \quad (53)$$

where the tetragonal symmetry tensor $T_{j_k r_k}$ was substituted for $\partial z_{r_k}/\partial s_{j_k}$. The same relationship is derived for the primed variables from Eqs. (48) and (50)

$$\partial^k c_p/\partial s'_{q_1} \ldots \partial s'_{q_k} = T_{pi}(\partial^k d_i/\partial z'_{r_1} \ldots \partial z'_{r_k}) T_{q_1 r_1} \ldots T_{q_k r_k} \quad (54)$$

$$p = i = 1, \ldots, 6; q_1 = \ldots = q_k = r_1 \ldots r_k = 1, \ldots, 12$$

In body coordinates, the 6 by 6 and 12 by 12 tetragonal symmetry tensors are composed of two or four matrices, such as in Eq. (31), respectively arrayed along the principle diagonal. The effect of the d-derivatives in Eqs. (53) and (54) is an exchange of the subscript pairs 2, 3; 5, 6; 8, 9; 11, 12, accompanied by a sign change. The value of the individual derivative remains unaltered. Thus, the derivatives $D_i^{j_1 \ldots j_k}$ which survive the first conditions are transformed into $C_p^{q_1 \ldots q_k}$ derivatives by Eqs. (53) and (54) and then subjected to the second condition, Eq. (52). If $\Sigma q_k + p + k + 1$ is an odd number, the particular derivative vanishes because of tetragonal symmetry. Note that the sign change of some of the components during the transfer from d to c derivatives is immaterial insofar as Eq. (52) is concerned; it is incurred by the same elements on both sides and therefore cancels.

Summary and Applications

The Principle of Material Indifference was the only physical law employed to derive the conditions for vanishing aerodynamic derivatives of aircraft and guided missiles. The other steps taken were mathematical artifices, which were used so that the final result could be stated in the simplest form possible.

To derive the condition of planar vehicles, two hypothetical flight tests were devised as mirror images. Because the external configuration of the vehicle exhibits reflectional symmetry, the airflow encounters the same shape in both tests; therefore, the dependence of the aerodynamic forces on the external configuration is the same and did not need to be stated explicitly. The dimension of the state vector thus was reduced to 12. The PMI for aerodynamic derivatives, Eq. (40), applied to the two tests, furnished the desired result, Eq. (46), after having first introduced sign functionals for compact notation.

Cruciform vehicles possess an additional plane of symmetry rotated by 90 deg. Besides the elimination process of planar vehicles, the aerodynamic derivatives were subjected to an additional condition derived from the second plane of symmetry. Two hypotethetical flight tests were constructed by 90 deg rotation from the original tests. Again, the two tests were mirror images, and the PMI yielded the second condition, Eq. (52). The derivatives in both eliminating equations are related by simply exchanging every second and third subscript.

For applications, a summary of the major results follows. Abbreviate the aerodynamic derivatives by

$$D_i^{j_1\cdots j_k} = (\partial^k d_i/\partial z_{j_1}\ldots\partial z_{j_k}) \tag{55}$$

with the force components in body axes

$$d_i = [X,Y,Z,L,M,N] \tag{56}$$

and the state vector in body axes

$$z_j = [u,v,w,p,q,r,\dot{u},\dot{v},\dot{w},\delta p,\delta q,\delta r] \tag{57}$$

The derivatives of vehicles with reflectional symmetry vanish if

$$\Sigma j_k + i + k + 1 = \text{Odd Number} \tag{58}$$

For vehicles with tetragonal symmetry, Eq. (58) eliminates only a portion of the derivatives. To formulate the second condition which will remove the remaining derivatives, introduce the auxiliary derivative $C_p^{q_1\cdots q_k}$ with the sequence of subscripts and superscripts rearranged according to Table 1.

The derivatives of vehicles with tetragonal symmetry vanish if either Eq. (58) or

$$\Sigma q_k + p + k + 1 = \text{Odd Number} \tag{59}$$

is satisfied. As a first example, consider the derivative of an aircraft

$$\partial^3 M/\partial v\partial w\partial\delta q = M_{vw\delta q} = D_5^{2\,3\,11} \tag{60}$$

Equation (58) yields $2+3+11+5+3+1 = \text{Odd}$; therefore, this derivative which expresses the linear pitch control effectiveness as a linear function of the cross velocities vanishes because of reflectional symmetry. However, the derivative that depends quadratically on v exists as shown

$$\partial^4 M/\partial v^2\partial w\partial\delta q = M_{v^2 w\delta q} = D_5^{2\,2\,3\,11} \tag{61}$$

$$2+2+3+11+5+4+1 = \text{Even Number} \tag{62}$$

As another example, the second order derivative

$$\partial^2 Y/\partial w\partial\delta r = Y_{w\delta r} = D_2^{3\,12} \tag{63}$$

survives the test for planar vehicles, meaning that the lateral force increment due to the rudder deflection is linearly dependent on the downwash effect. To check whether this derivative vanishes for cruciform vehicles, generate the auxiliary derivative

$$C_3^{2\,11} = D_2^{3\,12} \tag{64}$$

substituting the new subscripts and superscripts into Eq. (59) yields $2+11+3+2+1 = \text{Odd Number}$, that is, the derivative does not exist. Physically speaking, the downwash effect is symmetrical for cruciform configurations. It affects the side force not linearly, which would result in a sign change, but quadratically as shown

$$\partial^3 Y/\partial w^2\partial\delta r = D_2^{3\,3\,12} = C_3^{2\,2\,11} \tag{65}$$

Both derivatives survive the tests of Eqs. (58) and (59). Note that the sequence of taking partials can be rearranged without changing the value of the partial derivative, because the derivatives are assumed to be continuous.

For quick reference, Fig. 1 displays the aerodynamic derivatives, up to third order, which survive the reflectional symmetry test. Depending on the force components, i, the order of the derivative, and the even or odd integer of the third superscripts, the existence of the derivative is indicated by two symbols in the matrix arrays of Fig. 1. For instance, for the first order derivative X_u, the table of Fig. 1 assigns a cruciform symbol to the force component X. To determine existence, refer to the single row array. Because X_u is associated with a cruciform symbol, it exists. Other examples: X_v vanishes, but Y_v survives the test. For second order derivatives, the symbols are reversed, and the 12 by 12 array is used to determine existence. Accordingly, $Z_{w\delta q}$ exists, but $Y_{w\delta q}$ vanishes. Third order derivatives must be separated into two groups depending on an even or odd third order superscript (even or odd refers to the position number of the variable in the state vector). If this superscript is even; e.g., v, the cruciform symbol is associated with a derivative such as Z_{wrv}. The square array indicates existence of this derivative.

Table 1 Subscripts and superscripts

i, j_k	1	2	3	4	5	6	7	8	9	10	11	12
p, q_k	1	3	2	4	6	5	7	9	8	10	12	11

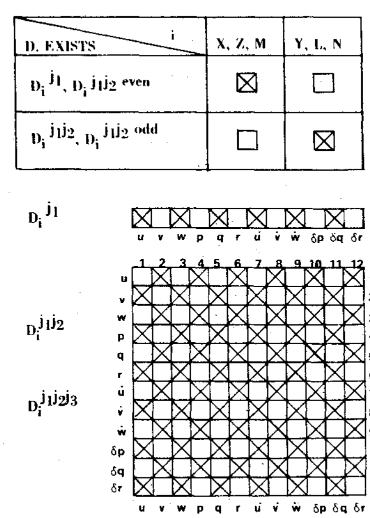

Fig. 1 Aerodynamic derivatives of planar vehicles.

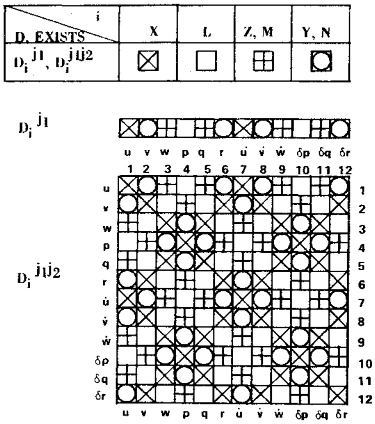

Fig. 2 Aerodynamic derivatives of cruciform vehicles.

For vehicles with tetragonal symmetry, a compact graphic display is possible only for first and second order derivatives. In Fig. 2, the table assigns the symbols to the force components for use in the arrays. For instance, X_u exists, and X_v vanishes; Z_{uw} survives, and $Z_{u\delta p}$ does not survive the tetragonal symmetry test.

The graphical aids of Figs. (1) and (2) can be used to uniquely determine the existence or nonexistence of aerodynamic derivatives. However, a derivative that passes the test still can have a negligible value and, therefore, can be set to zero. A significant number of derivatives can be eliminated by symmetry conditions alone. Reflectional symmetry eliminates half of the linear derivatives, and, because the square array is symmetrical, only approximately a quarter of the second and third order derivatives need be calculated. For vehicles with tetragonal symmetry, these numbers are further reduced by a factor of one-half.

The mathematical modeling of aerodynamic forces for computer simulations frequently includes tabular look-up for variables with large variations. The Taylor expansion is only carried out for those variables which remain small. So far, the results in this report have been stated in terms of complete expansions of all 12 components of the state vector. With minor modifications, the results are applicable to incomplete expansions. For instance, if the aerodynamic forces should be expressed as tabular functions of the velocity component u, The Taylor expansion is carried out in terms of the state variable components 2 through 12 only. All derivatives are functions of u. In applying the symmetry conditions, the derivatives are reduced in order by the number of partials in u, and they are made functions of u. Equal derivatives are combined, for example, for planar vehicles

$$X_{uw\delta q} + X_{u^2 w\delta q} + X_{u^3 w\delta q} = X_{w\delta q}\{u\} \tag{66}$$

This procedure applies to any derivative and any state variable component. Also, more than one variable can be replaced by implicit functionals.

The aerodynamic derivatives discussed in this report have not been nondimensionalized; rather, they are implicit functions of the state of the fluid and the shape of the vehicle. For computer applications, nondimensional derivatives are not required. However, it is convenient to substitute the Mach and Reynolds numbers for these implicit dependencies, while preserving the dimensions of the state variables in the partial derivatives.

The numerical values of the derivatives are determined from analytical analysis and wind tunnel testing. Conventional testing expresses the functional dependence on Mach number, side-slip angle, and angle-of-attack rather than u, v, w. The test results must first be converted into state variables before the derivatives can be evaluated. Alternately, for sufficiently small variables, the derivatives can be converted to conventional variables, and the wind tunnel results can be employed directly.

Recently, Lusardi et al.[5] pointed out that the rotational symmetry assumption is not valid to explain certain test results of spinning re-entry cones. The boundary-layer displacement, in effect, causes the flow to form a mirror symmetrical rather than a rotational pattern. Consequently, the simplifications of the Maple-Synge[1] theory cannot readily be made. The Magnus moment, in particular, need to be formulated for the less restrictive case of mirror symmetry. The techniques developed in this report permit this extension. The effect of unsymmetrical boundary-layer build-up on nonlinear derivatives can be investigated by comparison of the existence requirements for derivatives.

Conclusions

The theory of aerodynamic symmetry, as presented in this report, is a new approach to the problem of eliminating aerodynamic derivatives based on the configurational symmetries of aircraft and guided missiles. It is derived from the universal Principle of Material Indifference employing tensor analysis. It exceeds the classical Maple-Synge Theory because it also applies to unsteady aerodynamics and control effectiveness. The results are stated by two simple conditions, easily applied by the practicing aerodynamicist. They should be helpful in analyzing nonlinear aerodynamic effects and in accurately modeling aerodynamic forces for computer simulations.

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